

Limits of level and parameter dependent subdivision schemes: a matrix approach

Maria Charina^{a,*}, Costanza Conti^b, Nicola Guglielmi^c, Vladimir Protasov^d

^a*Fakultät Mathematik, Universität Wien, Austria*

^b*DIEF, University of Firenze, Italy*

^c*INDIRIZZO, University of L'Aquila, Italy*

^d*Moscow State University, Russia*

Abstract

In this paper, we present a new matrix approach for the analysis of subdivision schemes whose non-stationarity is due to linear dependency on parameters whose values vary in a compact set. Indeed, we show how to check the convergence in $C^\ell(\mathbb{R}^s)$ and determine the Hölder regularity of such level and parameter dependent schemes efficiently via the joint spectral radius approach. The efficiency of this method and the important role of the parameter dependency are demonstrated on several examples of subdivision schemes whose properties improve the properties of the corresponding stationary schemes. Moreover, we derive necessary criteria for a function to be generated by some level dependent scheme and, thus, expose the limitations of such schemes.

Keywords: Level dependent (non-stationary) subdivision schemes, tension parameter, sum rules, Hölder regularity, joint spectral radius.

1. Introduction

We analyze convergence and Hölder regularity of multivariate level dependent (non-stationary) subdivision schemes whose masks depend linearly on one or several parameters. For this type of schemes, which include well-known schemes with tension parameters [1, 2, 15, 17, 37, 38], the theoretical results from [8] are applicable, but not always efficient. Indeed, if the level dependent parameters vary in some compact set, then the set of the so-called limit points (see [8]) of the corresponding sequence of non-stationary masks exists, but cannot be determined explicitly. This hinders the regularity analysis of such schemes. Thus, we present a different perspective on the results in [8] and derive a new general method for convergence and regularity analysis of such level and parameter dependent schemes. The practical efficiency of this new method is illustrated on several examples. We also derive necessary criteria that allow us to describe the class of functions that can be generated by non-stationary subdivision schemes. Indeed, we show how to characterize such functions by the special property of the zeros of their Fourier transforms.

Subdivision schemes are iterative algorithms for generating curves and surfaces from given control points of a mesh. They are easy to implement and intuitive in use. These and other nice mathematical properties of subdivision schemes motivate their popularity in applications, i.e. in modelling of freeform curves and surfaces, approximation and interpolation of functions, computer animation, signal and image processing etc. Non-stationary subdivision schemes extend the variety of different shapes generated by stationary subdivision. Indeed, the level dependency enables to generate new classes of functions such as exponential polynomials, exponential B-splines, etc. This gives a new impulse to development of subdivision schemes

*Corresponding author

Email addresses: maria.charina@univie.ac.at (Maria Charina), costanza.conti@unifi.it (Costanza Conti), guglielm@units.it (Nicola Guglielmi), v-protasov@yandex.ru (Vladimir Protasov)

and enlarges the scope of their applications, e.g. in biological imaging [23, 43], geometric design [40, 42] or isogeometric analysis [3, 12].

The main challenges in the analysis of any subdivision scheme are its convergence (in various function spaces), the regularity of its limit functions and its generation and reproduction properties. The important role of the matrix approach for regularity analysis of stationary subdivision schemes is well-known. It allows to reduce the analysis to the computation or estimation of the joint spectral radius of the finite set of square matrices derived from the subdivision mask. Recent advances in the joint spectral radius computation [30, 39] makes the matrix approach very precise and efficient. In the non-stationary setting, however, this approach has never been applied because of the several natural obstacles. First of all, the matrix products that emerge in the study of non-stationary schemes have a different form than those usually analyzed by the joint spectral radius techniques. Secondly, the masks of non-stationary schemes do not necessarily satisfy sum rules, which destroys the relation between the convergence of the scheme and spectral properties of its transition matrices. All those difficulties were put aside by the results in [8], where the matrix approach was extended to general non-stationary setting.

In this paper, in Section 3, we make the next step and consider level and parameter dependent subdivision schemes whose masks include tension parameters, used to control the properties of the subdivision limit. Mostly, the tension parameters are level dependent and influence the asymptotic behavior of the scheme. If this is the case, the scheme can be analyzed by [8, Theorem 2], which states that the convergence and Hölder regularity of any such non-stationary scheme depends on the joint spectral radius of the matrices generated by the so-called limit points of the sequence of level-dependent masks. In Theorem 3.5, we show that for the schemes with linear dependence on these parameters, the result of [8, Theorem 2] can be simplified and be made more practical, see examples in Section 3.1. In Section 4, we address the problem of reproduction property of subdivision schemes and of characterizing the functions that can be generated by non-stationary subdivision schemes. This question is crucial in many aspects. For instance, the reproduction of exponential polynomials is strictly connected to the approximation order of a subdivision scheme and to its regularity [19]. Essentially, the higher is the number of exponential polynomials that are being reproduced, the higher is the approximation order and the possible regularity of the corresponding scheme.

2. Background

Let $M = mI \in \mathbb{Z}^{s \times s}$, $|m| \geq 2$, be a dilation matrix and $E = \{0, \dots, |m| - 1\}^s$ be the set of the coset representatives of $\mathbb{Z}^s / M\mathbb{Z}^s$. We study subdivision schemes given by the sequence $\{S_{\mathbf{a}(r)}, r \in \mathbb{N}\}$ of subdivision operators $S_{\mathbf{a}(r)} : \ell(\mathbb{Z}^s) \rightarrow \ell(\mathbb{Z}^s)$ that define the subdivision rules by

$$(S_{\mathbf{a}(r)} \mathbf{c})(\alpha) = \sum_{\beta \in \mathbb{Z}^s} \mathbf{a}_{\alpha - M\beta}^{(r)} c(\beta), \quad \alpha \in \mathbb{Z}^s.$$

The masks $\mathbf{a}^{(r)} = \{\mathbf{a}_{\alpha}^{(r)}, \alpha \in \mathbb{Z}^s\}$, $r \in \mathbb{N}$, are sequences of real numbers $\mathbf{a}_{\alpha}^{(r)}$ and are assumed to be all supported in $\{0, \dots, N\}^s$, $N \in \mathbb{N}$. For the given set

$$K = \sum_{r=1}^{\infty} M^{-1} G, \quad G = \{-|m|, \dots, N+1\}^s, \quad (2.1)$$

the masks define the square matrices

$$A_{\varepsilon}^{(r)} = \left(\mathbf{a}_{M\alpha + \varepsilon - \beta}^{(r)} \right)_{\alpha, \beta \in K}, \quad r \in \mathbb{N}, \quad \varepsilon \in E. \quad (2.2)$$

We assume that the level dependent symbols

$$a^{(r)}(z) = \sum_{\alpha \in \mathbb{Z}^s} \mathbf{a}_{\alpha}^{(r)} z^{\alpha}, \quad z^{\alpha} = z_1^{\alpha_1} \cdot \dots \cdot z_s^{\alpha_s}, \quad z \in (\mathbb{C} \setminus \{0\})^s.$$

of the subdivision scheme

$$c^{(r+1)} = S_{\mathbf{a}^{(r)}} c^{(r)} = S_{\mathbf{a}^{(r)}} S_{\mathbf{a}^{(r-1)}} \dots S_{\mathbf{a}^{(1)}} c^{(1)}, \quad r \in \mathbb{N},$$

satisfy sum rules of order $\ell + 1$, $\ell \in \mathbb{N}_0$. For more details on sum rules see e.g. [4, 5, 32, 34].

Definition 2.1. Let $\ell \in \mathbb{N}_0$, $r \in \mathbb{N}$. The symbol $a^{(r)}(z)$, $z \in (\mathbb{C} \setminus \{0\})^s$, satisfies sum rules of order $\ell + 1$ if

$$a^{(r)}(1, \dots, 1) = |m|^s \quad \text{and} \quad \max_{|\eta| \leq \ell} \max_{\epsilon \in \Xi \setminus \{1\}} |D^\eta a^{(r)}(\epsilon)| = 0, \quad (2.3)$$

where $\Xi = \{e^{-i\frac{2\pi}{|m|}\varepsilon} = (e^{-i\frac{2\pi}{|m|}\varepsilon_1}, \dots, e^{-i\frac{2\pi}{|m|}\varepsilon_s}), \varepsilon \in E\}$ and $D^\eta = \frac{\partial^{\eta_1}}{\partial z_1^{\eta_1}} \dots \frac{\partial^{\eta_s}}{\partial z_s^{\eta_s}}$.

The assumption that all symbols $a^{(r)}(z)$ satisfy sum rules of order $\ell + 1$, guarantees that the matrices $A_\varepsilon^{(r)}$, $\varepsilon \in E$, $r \in \mathbb{N}$, in (2.2) have common left-eigenvectors of the form

$$(p(\alpha))_{\alpha \in K}, \quad p \in \Pi_\ell,$$

where Π_ℓ is the space of polynomials of degree less than or equal to ℓ . Thus, the matrices $A_\varepsilon^{(r)}$, $\varepsilon \in E$, $r \in \mathbb{N}$, possess a common linear subspace $V_\ell \subset \mathbb{R}^{|K|}$ orthogonal to the span of the common left-eigenvectors of $A_\varepsilon^{(r)}$, $\varepsilon \in E$, $r \in \mathbb{N}$. The spectral properties of the set

$$\mathcal{T} = \{A_\varepsilon^{(r)}|_{V_\ell}, \varepsilon \in E, r \in \mathbb{N}\}$$

determine the regularity of the non-stationary scheme, see [8].

Remark 2.2. In the univariate case, i.e. $|m| = |M|$, the assumption that the symbols $a^{(r)}(z)$, $r \in \mathbb{N}$, satisfy sum rules of order $\ell + 1$ implies that

$$a^{(r)}(z) = (1 + z + \dots + z^{|m|-1})^\ell \sum_{\alpha \in \mathbb{Z}} b_\alpha^{(r)} z^\alpha, \quad z \in \mathbb{C} \setminus \{0\},$$

and

$$A_\varepsilon^{(r)}|_{V_\ell} = \left(b_{M\alpha + \varepsilon - \beta}^{(r)} \right)_{\alpha, \beta \in \{0, \dots, N-\ell\}}, \quad \varepsilon \in E. \quad (2.4)$$

In the multivariate case, the explicit form of the matrices $A_\varepsilon^{(r)}|_{V_\ell}$, $\varepsilon \in E$, $r \in \mathbb{N}$, depends on the choice of the basis of V_ℓ , see e.g. [8, Section 3.1] or [4].

Definition 2.3. A subdivision scheme $\{S_{\mathbf{a}^{(r)}}, r \in \mathbb{N}\}$ is C^ℓ -convergent, if for any initial sequence $\mathbf{c} \in \ell_\infty(\mathbb{Z}^s)$ there exists the limit function $g_{\mathbf{c}} \in C^\ell(\mathbb{R}^s)$ such that for any test function $f \in C^\ell(\mathbb{R}^s)$

$$\lim_{k \rightarrow \infty} \left\| g_{\mathbf{c}}(\cdot) - \sum_{\alpha \in \mathbb{Z}^s} S_{\mathbf{a}^{(r)}} S_{\mathbf{a}^{(r-1)}} \dots S_{\mathbf{a}^{(1)}} c(\alpha) f(M^k \cdot -\alpha) \right\|_{C^\ell} = 0. \quad (2.5)$$

For more details on test functions see [21]. Note that, it suffices to check (2.5) for only one test function f . Note also that, if all limits of a subdivision scheme belong to $C^\ell(\mathbb{R}^s)$, then the scheme may not converge in C^ℓ , but only in $C^0(\mathbb{R}^s)$.

In this paper, we also show how to estimate the Hölder regularity of subdivision limits.

Definition 2.4. The Hölder regularity of the C^0 -convergent scheme $\{S_{\mathbf{a}^{(r)}}, r \in \mathbb{N}\}$ is $\alpha = \ell + \zeta$, if ℓ is the largest integer such that $g_{\mathbf{c}} \in C^\ell(\mathbb{R}^s)$ and ζ is the supremum of $\nu \in [0, 1]$ such that

$$\max_{\mu \in \mathbb{N}_0^s, |\mu| = \ell} |D^\mu g_{\mathbf{c}}(x) - D^\mu g_{\mathbf{c}}(y)| \leq |x - y|^\nu, \quad x, y \in \mathbb{R}^s.$$

We call α the Hölder exponent of $\{S_{\mathbf{a}^{(r)}}, r \in \mathbb{N}\}$.

The joint spectral radius of a set of square matrices was introduced in [41] and is independent of the choice of the matrix norm $\|\cdot\|$.

Definition 2.5. *The joint spectral radius (JSR) of a compact family \mathcal{M} of square matrices is defined by*

$$\rho(\mathcal{M}) := \lim_{n \rightarrow \infty} \max_{M_1, \dots, M_n \in \mathcal{M}} \left\| \prod_{j=1}^n M_j \right\|^{1/n}.$$

The link between the JSR and subdivision is well-known, see e.g. [6, 11, 22, 33, 35].

3. Parameter dependent subdivision schemes: matrix approach

There are several examples of subdivision schemes that include a tension parameter. We call them parameter dependent schemes. Often the tension parameter is level dependent and shows a certain asymptotic behavior which implies the asymptotic behavior of the corresponding non-stationary scheme, i.e. $\lim_{r \rightarrow \infty} \mathbf{a}^{(r)} = \mathbf{a}$.

In this case, although the set $\{A_\varepsilon^{(r)}, \varepsilon \in E, r \in \mathbb{N}\}$ is not compact, the convergence and regularity of the scheme $\{S_{\mathbf{a}^{(r)}}, r \in \mathbb{N}\}$ can be analyzed via the joint spectral radius approach in [8]. The results in [8] are still applicable even if the parameter values vary in some compact interval. Indeed, the existence of the limit points for the sequence $\{\mathbf{a}^{(r)}, r \in \mathbb{N}\}$ of the subdivision masks is guaranteed, though these limit points are not always explicitly known.

Definition 3.1. *For the mask sequence $\{\mathbf{a}^{(r)}, r \in \mathbb{N}\}$ we denote by \mathcal{A} the set of its limit points, i.e. the set of masks \mathbf{a} such that*

$$\mathbf{a} \in \mathcal{A}, \quad \text{if } \exists \{r_n, n \in \mathbb{N}\} \text{ such that } \lim_{n \rightarrow \infty} \mathbf{a}^{(r_n)} = \mathbf{a}.$$

In this section, we show that the joint spectral radius approach can be effectively applied even if the limit points of $\{\mathbf{a}^{(r)}, r \in \mathbb{N}\}$ cannot be determined explicitly, but the masks $\mathbf{a}^{(r)}$ depend linearly on the parameter $\omega^{(r)} \in [\omega_1, \omega_2]$, $-\infty < \omega_1 < \omega_2 < \infty$.

Well-known and celebrated examples of parameter dependent stationary subdivision schemes with linear dependence on the parameter are e.g. the univariate four point scheme [25] with the symbol

$$a(z, \omega) = \frac{(1+z)^2}{2} + \omega(-z^{-2} + 1 + z^2 - z^4), \quad \omega \in \left[0, \frac{1}{16}\right], \quad z \in \mathbb{C} \setminus \{0\},$$

which is a parameter perturbation of the linear B-spline. Also the bivariate butterfly scheme [26] with the symbol

$$a(z_1, z_2, \omega) = \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2) + \omega c(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

with

$$\begin{aligned} c(z_1, z_2) = & z_1^{-1}z_2^{-2} + z_2^2z_1^{-1} + z_1^{-2}z_2^{-1} + z_1^2z_2^{-1} - 2z_1^2z_2^3 - 2z_1^3z_2^2 + z_1^2z_2^4 + z_1^4z_2^2 + z_1^3z_2^4 \\ & + z_1^4z_2^3 - 2z_1^{-1} + z_1^{-2} - 2z_1^2 - 2z_2^{-1} + z_1^3 + z_2^{-2} - 2z_2^2 + z_2^3 \end{aligned} \quad (3.1)$$

is a parameter perturbation of the linear three-directional box spline. Other examples of such parameter dependent schemes are those with symbols that are convex combinations

$$\omega a(z_1, z_2) + (1-\omega) b(z_1, z_2) = b(z_1, z_2) + \omega (a(z_1, z_2) - b(z_1, z_2)), \quad \omega \in [0, 1], \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

of two (or more) symbols of stationary schemes, see e.g. [7, 14, 16, 29]. Known are also their non-stationary univariate counterparts with level dependent parameters $\omega^{(r)}$ (see [1, 2, 15, 17], for example)

$$\frac{(1+z)^2}{2} + \omega^{(r)}(-z^{-2} + 1 + z^2 - z^4), \quad r \in \mathbb{N}, \quad \lim_{r \rightarrow \infty} \omega^{(r)} = \omega \in \mathbb{R},$$

and

$$\omega^{(r)} a(z) + (1 - \omega^{(r)}) b(z), \quad r \in \mathbb{N}, \quad \omega^{(r)} \in [0, 1].$$

Note that the use of the level dependent parameters sometimes allows us to enhance the properties of the existing stationary schemes (e.g. with respect to their smoothness, size of their support or reproduction and generation properties [7, 14, 15, 17]).

In all schemes considered above, the subdivision rules depend either on the same, fixed, parameter $\omega = \omega^{(r)} \in [\omega_1, \omega_2]$ independent of r , or the parameters $\omega^{(r)} \in [\omega_1, \omega_2]$ are chosen in a such a way that either $\lim_{r \rightarrow \infty} \omega^{(r)} = \omega \in [\omega_1, \omega_2]$ or the corresponding non-stationary scheme is asymptotically equivalent to some known stationary scheme. In this section, we provide a matrix method for analyzing regularity of more general subdivision schemes: we consider the level dependent masks $\mathbf{a}(\omega^{(r)}) = \{a_\alpha(\omega^{(r)}), \alpha \in \mathbb{Z}^s\}$, $r \in \mathbb{N}$, and require that $\omega^{(r)} \in [\omega_1, \omega_2]$ without any further assumptions on the behavior of the sequence $\{\omega^{(r)}, r \in \mathbb{N}\}$. We assume, however, that each of the masks depends linearly on the corresponding parameter $\omega^{(r)}$.

The level dependent masks $\{\mathbf{a}(\omega^{(r)}), r \in \mathbb{N}\}$ define the corresponding square matrices which we denote by

$$A_{\varepsilon, \omega^{(r)}} = \left(a_{M\alpha + \varepsilon - \beta}(\omega^{(r)}) \right)_{\alpha, \beta \in K}, \quad \varepsilon \in E, \quad (3.2)$$

and the level dependent symbols

$$a(z, \omega^{(r)}) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha(\omega^{(r)}) z^\alpha, \quad z^\alpha = z_1^{\alpha_1} \cdots z_s^{\alpha_s}, \quad z \in (\mathbb{C} \setminus \{0\})^s.$$

The assumption that each mask $\mathbf{a}(\omega^{(r)})$ depends linearly on $\omega^{(r)}$, leads to the following immediate, but crucial result.

Proposition 3.2. *Let $\ell \in \mathbb{N}_0$ and $-\infty < \omega_1 < \omega_2 < \infty$. If every symbol of the sequence $\{a(z, \omega^{(r)}), r \in \mathbb{N}\}$ depends linearly on the parameter $\omega^{(r)} \in [\omega_1, \omega_2]$ and satisfies sum rules of order $\ell + 1$, then every matrix in $\mathcal{T} = \{A_{\varepsilon, \omega^{(r)}}|_{V_\ell}, \omega^{(r)} \in [\omega_1, \omega_2], \varepsilon \in E, r \in \mathbb{N}\}$ is a convex combination of the matrices with $\omega^{(r)} \in \{\omega_1, \omega_2\}$*

$$A_{\varepsilon, \omega^{(r)}}|_{V_\ell} = (1 - t^{(r)})A_{\varepsilon, \omega_1}|_{V_\ell} + t^{(r)}A_{\varepsilon, \omega_2}|_{V_\ell}, \quad t^{(r)} \in [0, 1].$$

Proof. Let $r \in \mathbb{N}$. We first write $\omega^{(r)}$ as a convex combination of ω_1 and ω_2 , i.e.

$$\omega^{(r)} = (1 - t^{(r)})\omega_1 + t^{(r)}\omega_2 \quad \text{with} \quad t^{(r)} \in [0, 1].$$

Note that all entries of the matrices $A_{\varepsilon, \omega^{(r)}}, \varepsilon \in E$, are the coefficients of the corresponding mask $\mathbf{a}(\omega^{(r)})$. Since the mask coefficients depend linearly on the parameter $\omega^{(r)}$, so do the matrices $A_{\varepsilon, \omega^{(r)}}$, and hence, the corresponding linear operators. Therefore, the restrictions of these operators to their common invariant subspace V_ℓ also depend linearly on this parameter. \square

In the level independent case, i.e. $\omega^{(r)} = \omega$ for all $r \in \mathbb{N}$, the use of the joint spectral radius approach for studying the convergence and regularity of the corresponding stationary subdivision schemes is well understood. To show how this approach can be applied in the our non-stationary setting, we need first to prove the following auxiliary result.

Proposition 3.3. *Let $\ell \in \mathbb{N}_0$ and*

$$\mathcal{T} = \{A_{\varepsilon, \omega^{(r)}}|_{V_\ell}, \omega^{(r)} \in [\omega_1, \omega_2], \varepsilon \in E, r \in \mathbb{N}\} \quad (3.3)$$

be the infinite family of square matrices. If the JSR of the family $\mathcal{T}_{\omega_1, \omega_2} = \{A_{\varepsilon, \omega_1}|_{V_\ell}, A_{\varepsilon, \omega_2}|_{V_\ell}, \varepsilon \in E\}$ satisfies $\rho(\mathcal{T}_{\omega_1, \omega_2}) = \gamma$, then $\rho(\mathcal{T}) = \gamma$.

Proof. First of all observe that $\rho(\mathcal{T}_{\omega_1, \omega_2}) = \gamma$ implies, for any $\delta > 0$, the existence of a δ -extremal norm (see e.g. [27, 31]), i.e. an operator norm $\|\cdot\|_\delta$ such that

$$\|A_{\varepsilon, \omega_1}|_{V_\ell}\|_\delta \leq \gamma + \delta, \quad \|A_{\varepsilon, \omega_2}|_{V_\ell}\|_\delta \leq \gamma + \delta. \quad (3.4)$$

Then, by Proposition 3.2, estimates in (3.4) and subadditivity of matrix operator norms, we get

$$\|A_{\varepsilon, \omega^{(r)}}|_{V_\ell}\|_\delta = \|(1-t^{(r)})A_{\varepsilon, \omega_1}|_{V_\ell} + t^{(r)}A_{\varepsilon, \omega_2}|_{V_\ell}\|_\delta \leq (1-t^{(r)})\|A_{\varepsilon, \omega_1}|_{V_\ell}\|_\delta + t^{(r)}\|A_{\varepsilon, \omega_2}|_{V_\ell}\|_\delta = \gamma + \delta, \quad t^{(r)} \in [0, 1].$$

This, due to the arbitrary choice of $\delta > 0$, implies that $\rho(\mathcal{T}) = \gamma$, which concludes the proof. \square

Remark 3.4. (i) Note that, if the family $\mathcal{T}_{\omega_1, \omega_2}$ is non-defective, i.e. there exists an extremal norm $\|\cdot\|$ such that $\max_{\varepsilon \in E} \{\|A_{\varepsilon, \omega_1}|_{V_\ell}\|, \|A_{\varepsilon, \omega_2}|_{V_\ell}\|\} = \gamma$, then \mathcal{T} is also non-defective and all products of degree d of the associated product semigroup have maximal growth bounded by γ^d . Note also that for any family of matrices \mathcal{B} , $\mathcal{B} \subset \mathcal{T}$, it follows that $\rho(\mathcal{B}) \leq \gamma$. (ii) Moreover, if a family \mathcal{T} is irreducible, i.e., its matrices do not have a common nontrivial subspace, then \mathcal{T} is non-defective. Therefore, the case of non-defective families is quite general.

We are now ready to formulate the main result of this section.

Theorem 3.5. Let $\ell \in \mathbb{N}_0$. Assume that every symbol of the sequence $\{a(z, \omega^{(r)}), r \in \mathbb{N}\}$ depends linearly on $\omega^{(r)} \in [\omega_1, \omega_2]$ and satisfies sum rules of order $\ell + 1$. Then the non-stationary scheme $\{S_{\mathbf{a}(\omega^{(r)})}, r \in \mathbb{N}\}$ is C^ℓ -convergent, if the JSR of the family $\mathcal{T}_{\omega_1, \omega_2} = \{A_{\varepsilon, \omega_1}|_{V_\ell}, A_{\varepsilon, \omega_2}|_{V_\ell}, \varepsilon \in E\}$ satisfies

$$\rho(\mathcal{T}_{\omega_1, \omega_2}) = \gamma < |m|^{-\ell}. \quad (3.5)$$

Moreover the Hölder exponent of its limit functions is $\alpha \geq -\log_{|m|} \gamma$.

Proof. Since the parameters $\{\omega^{(r)}, r \in \mathbb{N}\}$ vary in the compact interval $[\omega_1, \omega_2]$, there exists a set of limits points (finite or infinite) for the sequence $\{\mathbf{a}(\omega^{(r)}), r \in \mathbb{N}\}$ of subdivision masks. Let us denote this set by \mathcal{A} and the corresponding set of square matrices by $\mathcal{T}_{\mathcal{A}} = \{A_\varepsilon = (a_{M\alpha + \varepsilon - \beta})_{\alpha, \beta \in K}, \varepsilon \in E, \mathbf{a} \in \mathcal{A}\}$. Obviously, $\mathcal{T}_{\mathcal{A}} \subset \mathcal{T}$ with \mathcal{T} as in (3.3). Since by Proposition 3.3 and Remark 3.4, $\rho(\mathcal{T}_{\mathcal{A}}) \leq \gamma$, the claim follows by [8, Corollary 1]. \square

Remark 3.6. (i) Note that, due to $\rho(\mathcal{T}_{\mathcal{A}}) \leq \gamma$, Theorem 3.5 yields a smaller Hölder exponent α than what could be obtained by [8, Corollary 1]. For example, consider binary subdivision scheme with the symbols

$$\begin{aligned} a(z, \omega^{(r)}) &= z^{-1} \frac{(1+z)^2}{2}, \quad r \in \{1, \dots, L\}, \quad L \in \mathbb{N}, \\ a(z, \omega^{(r)}) &= z^{-1} \frac{(1+z)^2}{2} + \frac{1}{16}(-z^{-3} + z^{-1} + z - z^3), \quad r \geq L+1, \quad z \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

To apply Theorem 3.5, we can view the corresponding masks as being linearly dependent on parameters $\omega^{(r)} \in [0, \frac{1}{16}]$. The corresponding family $\mathcal{T}_{0, \frac{1}{16}} = \{A_{\varepsilon, 0}|_{V_1}, A_{\varepsilon, \frac{1}{16}}|_{V_1}, \varepsilon \in \{0, 1\}\}$ consists of the four matrices

$$A_{0, \omega}|_{V_1} = \begin{pmatrix} -\omega & -2\omega + \frac{1}{2} & -\omega & 0 \\ 0 & 2\omega & 2\omega & 0 \\ 0 & -\omega & -2\omega + \frac{1}{2} & -\omega \\ 0 & 0 & 2\omega & 2\omega \end{pmatrix}, \quad A_{1, \omega}|_{V_1} = \begin{pmatrix} 2\omega & 2\omega & 0 & 0 \\ -\omega & -2\omega + \frac{1}{2} & -\omega & 0 \\ 0 & 2\omega & 2\omega & 0 \\ 0 & -\omega & -2\omega + \frac{1}{2} & -\omega \end{pmatrix} \quad (3.6)$$

for $\omega \in \{0, \frac{1}{16}\}$. Due to

$$\max_{\varepsilon \in \{0, 1\}} \left\{ \|A_{\varepsilon, 0}|_{V_1}\|_\infty, \|A_{\varepsilon, \frac{1}{16}}|_{V_1}\|_\infty \right\} = \max_{\varepsilon \in \{0, 1\}} \left\{ \rho(A_{\varepsilon, 0}|_{V_1}), \rho(A_{\varepsilon, \frac{1}{16}}|_{V_1}) \right\} = \frac{1}{2},$$

we get $\rho(\mathcal{T}_{0, \frac{1}{16}}) = \frac{1}{2}$ and, thus, the corresponding scheme is convergent and has the Hölder exponent $\alpha \geq 1$. On the other hand, the set \mathcal{A} of limit points of the masks can be explicitly determined in this case and consists of the mask of the four point scheme. Thus, by [8, Corollary 1], the Hölder exponent is actually $\alpha \geq 2$.

(ii) The regularity estimate given in Theorem 3.5 can be improved, if the actual range of the parameters $\omega^{(r)}$, $r \geq L$, for some $L \in \mathbb{N}$, is a subinterval of $[\omega_1, \omega_2]$, see section 3.1.

(iii) Note that the result of Theorem 3.5 is directly extendable to the case when the matrix family \mathcal{T} depends linearly on a convex polyhedral set $\Omega = \text{co}\{\omega_1, \dots, \omega_L\}$ of parameters $\omega^{(r)} \in \Omega \subset \mathbb{R}^p$, $r \in \mathbb{N}$, such that

$$\omega^{(r)} = \sum_{j=1}^L t_j^{(r)} \omega_j \quad \text{with} \quad t_j^{(r)} \in [0, 1] \quad \text{and} \quad \sum_{j=1}^L t_j^{(r)} = 1.$$

This is the case, for example, when we define the level and parameter dependent symbols

$$a(z, \omega^{(r)}) = \sum_{j=1}^p \omega_j^{(r)} a_j(z), \quad \omega^{(r)} = (\omega_1^{(r)}, \dots, \omega_p^{(r)})^T \in \Omega, \quad r \in \mathbb{N}.$$

3.1. Examples

In this section we present two univariate examples of level dependent parameter schemes, whose constructions are based on the four point and six point Dubuc-Deslauriers schemes. In particular, in Example 3.7, the non-stationary scheme is constructed in such a way that the support of its limit function

$$\phi_1 = \lim_{r \rightarrow \infty} S_{\mathbf{a}(\omega^{(r)})} \dots S_{\mathbf{a}(\omega^{(1)})} \delta, \quad \delta(\alpha) = \begin{cases} 1, & \alpha = 0, \\ 0, & \text{otherwise} \end{cases}, \quad \alpha \in \mathbb{Z}^s,$$

is smaller than the support of the four point scheme, but its regularity is comparable. In Example 3.8, every non-stationary mask is a convex combination of the four point and six point Dubuc-Deslauriers schemes. We show how the regularity of the corresponding non-stationary scheme depends on the range of the corresponding parameters $\{\omega^{(r)}, r \in \mathbb{N}\}$. Both examples illustrate the importance of the dependency on several parameters $\{\omega^{(r)}, r \in \mathbb{N}\}$ instead of one $\omega \in \mathbb{R}$.

Example 3.7. We consider the univariate, binary scheme with the symbols

$$\begin{aligned} a(z, \omega^{(r)}) &= z^{-1} \frac{(1+z)^2}{2}, \quad r \in \{1, 2\}, \\ a(z, \omega^{(r)}) &= z^{-1} \frac{(1+z)^2}{2} + \omega^{(r)}(-z^{-3} + z^{-1} + z - z^3), \quad r \geq 3, \quad z \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

where $\omega^{(r)}$ are chosen at random from the interval $[\frac{3}{64}, \frac{1}{16}]$. The corresponding family

$$\mathcal{T}_{0, \frac{1}{16}} = \{A_{\varepsilon, 0}|_{V_1}, A_{\varepsilon, \frac{1}{16}}|_{V_1}, \varepsilon \in \{0, 1\}\}$$

consists of the same four matrices as in (3.6). And at the first glance the Hölder exponent of this scheme is $\alpha \geq 1$. On the other hand, we can view this scheme as the one with the corresponding matrix family

$$\mathcal{T}_{\frac{3}{64}, \frac{1}{16}} = \{A_{\varepsilon, \frac{3}{64}}|_{V_1}, A_{\varepsilon, \frac{1}{16}}|_{V_1}, \varepsilon \in \{0, 1\}\},$$

applied to a different starting data. Then we get $\rho(\mathcal{T}_{\frac{3}{64}, \frac{1}{16}}) = 3/8$ and, by Theorem 3.5, the Hölder exponent is actually $\alpha \geq -\log_2 \frac{3}{8} \approx 1.4150$.

The size of the support of ϕ_1 can be determined using the technique from [13] and is given by

$$\left[\sum_{k=0}^{\infty} 2^{-k-1} \ell(k), \sum_{k=0}^{\infty} 2^{-k-1} r(k) \right] = \left[-\frac{3}{2}, \frac{3}{2} \right]$$

with

$$\begin{aligned}\ell(k) &= -1, & r(k) &= 1, & k &= 0, 1, \\ \ell(k) &= -3, & r(k) &= 3, & k &\geq 2.\end{aligned}$$

Recall that the support of the basic limit function of the four point scheme is $[-3, 3]$.

Example 3.8. In this example we consider the univariate non-stationary scheme with symbols

$$a(z, \omega^{(r)}) = \omega^{(r)}a(z) + (1 - \omega^{(r)})b(z), \quad \omega^{(r)} \in [0, 1], \quad z \in \mathbb{C} \setminus \{0\},$$

where

$$a(z) = -\frac{z^{-3}(z+1)^4}{16}(z^2 - 4z + 1)$$

is the symbol of the four point scheme and

$$b(z) = \frac{z^{-5}(z+1)^6}{256}(3z^4 - 18z^3 + 38z^2 - 18z + 3)$$

is the symbol of C^2 -convergent quintic Dubuc-Deslauriers subdivision scheme [24]. By [28], the Hölder exponent of the $S_{\mathbf{b}}$ is $\alpha \approx 2.8301$. To determine the regularity of this level and parameter dependent scheme we consider the matrix set

$$\mathcal{T}_{0,1} = \{A_{\varepsilon,0}|_{V_2}, A_{\varepsilon,1}|_{V_2}, \varepsilon \in \{0, 1\}\}$$

with the four matrices

$$\begin{aligned}A_{0,\omega}|_{V_2} &= \frac{1}{256} \begin{pmatrix} 3-3\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7-9\omega & -9+9\omega & 3-3\omega & 0 & 0 & 0 & 0 & 0 \\ 45+3\omega & 45+3\omega & -7-9\omega & -9+9\omega & 3-3\omega & 0 & 0 & 0 \\ -9+9\omega & -7-9\omega & 45+3\omega & 45+3\omega & -7-9\omega & -9+9\omega & 3-3\omega & 0 \\ 0 & 3-3\omega & -9+9\omega & -7-9\omega & 45+3\omega & 45+3\omega & -7-9\omega & 0 \\ 0 & 0 & 0 & 3-3\omega & -9+9\omega & -7-9\omega & 45+3\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 3-3\omega & -9+9\omega & 0 \end{pmatrix}, \\ A_{1,\omega}|_{V_2} &= \frac{1}{256} \begin{pmatrix} -9+9\omega & 3-3\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 45+3\omega & -7-9\omega & -9+9\omega & 3-3\omega & 0 & 0 & 0 & 0 \\ -7-9\omega & 45+3\omega & 45+3\omega & -7-9\omega & -9+9\omega & 3-3\omega & 0 & 0 \\ 3-3\omega & -9+9\omega & -7-9\omega & 45+3\omega & 45+3\omega & -7-9\omega & -9+9\omega & 0 \\ 0 & 0 & 3-3\omega & -9+9\omega & -7-9\omega & 45+3\omega & 45+3\omega & 0 \\ 0 & 0 & 0 & 0 & 3-3\omega & -9+9\omega & -7-9\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3-3\omega & 0 \end{pmatrix}\end{aligned}$$

for $\omega \in \{0, 1\}$. In this case, the regularity of the non-stationary scheme $\{S_{\mathbf{a}(\omega^{(r)})}, r \in \mathbb{N}\}$ coincides with the regularity of the four point scheme. For $\omega \in (a, 1]$, $a > 0$, the scheme $\{S_{\mathbf{a}(\omega^{(r)})}, r \in \mathbb{N}\}$ is C^2 -convergent. And, for $\omega \in \{0, a\}$, $a < 1$, extensive numerical experiments show that the JSR of the family $\mathcal{T}_{0,a}$ is determined by the subfamily $\{A_{\varepsilon,a}|_{V_2}, \varepsilon \in \{0, 1\}\}$. For example, for $a = \frac{1}{2}$, we obtain $\rho(\mathcal{T}_{0,\frac{1}{2}}) \approx 0.2078$ and, thus, the corresponding Hölder exponent is $\alpha \geq 2.2662$.

4. Limitations of generation properties of non-stationary schemes

It is known that certain level dependent (non-stationary) subdivision schemes are capable of generating/reproducing certain spaces of exponential polynomials, see e.g. [9, 18]. In this section, we are interested in answering the question: How big is the class of functions that can be generated/reproduced by such schemes?

More precisely, we show that, already in the univariate setting, the zero sets of the Fourier transforms of the limit functions

$$\phi_k = \lim_{r \rightarrow \infty} S_{\mathbf{a}^{(r)}} \dots S_{\mathbf{a}^{(k)}} \delta, \quad \delta(\alpha) = \begin{cases} 1, & \alpha = 0, \\ 0, & \text{otherwise} \end{cases}, \quad \alpha \in \mathbb{Z}^s,$$

of such schemes are unions of the sets

$$\Gamma_r = \{\omega \in \mathbb{C} : a^{(r)}(e^{-i2\pi M^{-r}\omega}) = 0\}, \quad r \geq k,$$

and that the sets Γ_r are such that $\Gamma_r + M^r \mathbb{Z} = \Gamma_r$. Thus, some elementary functions cannot be generated by non-stationary schemes, see Example 4.4. Also the requirement that

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x) e^{-i2\pi x \omega} dx, \quad \omega \in \mathbb{C}, \quad k \in \mathbb{N},$$

is an entire function, limits the properties of the functions that can be generated by non-stationary subdivision schemes.

Proposition 4.1. *Let $\{\phi_k, k \in \mathbb{N}\}$ be continuous functions of compact support satisfying*

$$\phi_k(x) = \sum_{\alpha \in \mathbb{Z}} a^{(k)}(\alpha) \phi_{k+1}(Mx - \alpha), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}.$$

Then

$$\{\omega \in \mathbb{C} : \hat{\phi}_k(\omega) = 0\} = \bigcup_{r \geq k} \Gamma_r,$$

such that the sets Γ_r satisfy

$$\Gamma_r + M^r \mathbb{Z} = \Gamma_r.$$

Proof. Let $k \in \mathbb{N}$. By Paley-Wiener theorem, the Fourier transform $\hat{\phi}_k$ defined on \mathbb{R} has an analytic extension

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x) e^{-i2\pi x \omega} dx, \quad \omega \in \mathbb{C},$$

to the whole complex plane \mathbb{C} and $\hat{\phi}_k$ is an entire function. By Weierstrass theorem [20], every entire function can be represented by a product involving its zeroes. Define the sets

$$\Gamma_r := \{\omega \in \mathbb{C} : a^{(r)}(e^{-i2\pi M^{-r}\omega}) = 0\}, \quad r \in \mathbb{N}.$$

Let $z_{r,1}, \dots, z_{r,N}$ be the zeros of the polynomials $a^{(r)}(e^{-i2\pi M^{-r}\omega})$, counting their multiplicities. Then

$$\Gamma_r = iM^r \bigcup_{\ell=1}^N \text{Ln}(z_{r,\ell}),$$

where, by the properties of the complex logarithm, each of the sets $iM^r \text{Ln}(z_{r,\ell})$ consists of sequences of complex numbers and is M^r -periodic. Thus, each of the sets Γ_r satisfy

$$\Gamma_r + M^r \mathbb{Z} = \Gamma_r, \quad r \in \mathbb{N}.$$

The definition of $\hat{\phi}_k$ as an infinite product of the trigonometric polynomials $a^{(r)}(e^{-i2\pi M^{-r}\omega})$, $r \geq k$, yields the claim. □

The following examples illustrate the result of Proposition 4.1.

Example 4.2. The basic limit function of the simplest stationary scheme is given by $\phi_1 = \chi_{[0,1]}$. Its Fourier transform is

$$\hat{\phi}_1(\omega) = \frac{1 - e^{-i2\pi\omega}}{i2\pi\omega}, \quad \text{and} \quad \{\omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0\} = \mathbb{Z} \setminus \{0\}.$$

The mask symbol $a(z) = 1 + z$ has a single zero at $z = -1$, i.e. $e^{-i2\pi 2^{-r}\omega} = -1$ for $\omega = 2^r\{\frac{1}{2} + k : k \in \mathbb{Z}\}$, $r \in \mathbb{N}_0$. In other words, $\Gamma_1 = \{1 + 2k : k \in \mathbb{Z}\}$ and $\Gamma_r = 2\Gamma_{r-1}$ for $r \geq 2$. Therefore,

$$\{\omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0\} = \bigcup_{r \in \mathbb{N}} \Gamma_r.$$

Example 4.3. The first basic limit function of the simplest non-stationary scheme is given by $\phi_1(x) = \chi_{[0,1)}(x)e^{\lambda x}$, $\lambda \in \mathbb{C}$. Its Fourier transform is

$$\hat{\phi}_1(\omega) = \frac{e^{-i2\pi\omega + \lambda} - 1}{-i2\pi\omega + \lambda}, \quad \omega \in \mathbb{C}, \quad \text{and} \quad \{\omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0\} = -\frac{i\lambda}{2\pi} + \mathbb{Z} \setminus \{0\}.$$

The mask symbol $a^{(r)}(z) = 1 + e^{\lambda 2^{-r}}z$ has a single zero at $z = -e^{-\lambda 2^{-r}}$, i.e. $e^{-i2\pi 2^{-r}\omega} = -e^{-\lambda 2^{-r}}$ for $\omega = -\frac{i\lambda}{2\pi} + 2^r\{\frac{1}{2} + k : k \in \mathbb{Z}\}$, $r \in \mathbb{N}$. Note that $\Gamma_1 = -\frac{i\lambda}{2\pi} + \{1 + 2k : k \in \mathbb{Z}\}$ and

$$\bigcup_{r \in \mathbb{N}} 2^r\{\frac{1}{2} + k : k \in \mathbb{Z}\} = \mathbb{Z} \setminus \{0\}.$$

Therefore,

$$\{\omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0\} = \bigcup_{r \in \mathbb{N}} \Gamma_r.$$

In the next example we identify a compactly supported function that cannot be generated by any non-stationary subdivision scheme.

Example 4.4. Let us consider the compactly supported function

$$f(x) = \chi_{[-1,1]}(x) \frac{2}{\sqrt{1-x^2}}, \quad x \in \mathbb{R}.$$

It cannot be a limit of any non-stationary subdivision scheme. Indeed, its Fourier transform

$$J_0(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx, \quad \omega \in \mathbb{C}, \quad (4.1)$$

is the Bessel function J_0 of the first kind, which is entire, but has only positive zeros. The lower bound for its zeros $j_{0,s}$, $s \in \mathbb{N}$, is given by $j_{0,s} > \sqrt{(s - \frac{1}{4})^2 \pi^2}$, see [36]. Thus, Proposition 4.1 implies the claim.

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